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Pricing Currency Options in the Heston/CIR Double Exponential Jump-Diffusion Model

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Abstract

We examine currency options in the double exponential jump-diffusion version of the Heston stochastic volatility model for the exchange rate. We assume, in addition, that the domestic and foreign stochastic interest rates are governed by the CIR dynamics. The instantaneous volatility is correlated with the dynamics of the exchange rate return, whereas the domestic and foreign short-term rates are assumed to be independent of the dynamics of the exchange rate and its volatility. The main result furnishes a semi-analytical formula for the price of the European currency call option in the hybrid foreign exchange/interest rates model.

JEL Classification: G13

1 Introduction

We extend the results from Ahlip and Rutkowski (2015) by deriving a semi-analytical pricing formula for the currency option in a model where the volatility of the spot exchange rate is specified by the extended Heston model to include double exponential jumps considered by Kou(2002), Kou *et al.*

(2004) and Ning Cai *et al.*(2011) whereas the domestic and foreign interest rates are governed by the Cox–Ingersoll–Ross (CIR) dynamics postulated in Cox *et al.* (1985). In particular, the model put forward in the present work allows for a non-zero correlation between the exchange rate process and its instantaneous volatility. According to the model given by (1), the CIR interest rate processes are independent of one another, and they are also independent of the foreign exchange rate and its volatility, which in turn is jointly governed by a double exponential jump process, an extension of Heston's model.

In the seminal paper by Heston (1993), the author noted that increasing the volatility of volatility only increases the kurtosis of spot returns and does not capture skewness. In order to capture the skewness, it is crucial to include also the properly specified correlation between the volatility and the spot exchange rate returns. However, the Heston model is not always able to fit the implied volatility smile very well, particularly at short maturities (Gatheral, 2006). Christofferson *et al.*(2009), considered a two-factor structure for the volatility in their double Heston volatility model to enrich the variance process. For a detailed analysis the interested reader is referred to, for instance, Rouah (2013).

In papers by Bakshi *et al.* (1997), Bates (1996), and Duffie *et al.* (2000), the authors showed that stochastic volatility models do not offer reliable prices for close to expiration derivatives. This motivated Bates (1996) and Bakshi *et al.* (1997) to introduce jumps to the dynamics of the underlying. However, as observed by Andersen and Andreasen (2000) and Alizadeh *et al.* (2002), the addition of jumps to the dynamics of the underlying is not sufficient to capture the sudden increase in volatility due to market turbulence. Since the overall volatility in financial markets consists of a highly persistent slow moving and a rapid moving component, Eraker *et al.* (2003) proposed to introduce a jump process to the dynamics of the volatility process in order to enhance the cross-sectional impact on option prices.

Though jumps have very little effect on the shape of the volatility surface for long-dated options, Gatheral (2006) observed a more significant aspect as to why we consider jumps. The impact on the shape of the volatility surface is all at the short-expiration end, which further might explain why the skew is so steep for very short expirations and why the very short-dated term structure of skew is inconsistent with any stochastic volatility model.

More recently, D'Ippoliti *et al.* (2010) obtained closed-form solutions, in the spirit of Heston, for a model with jumps in both spot returns of the underlying asset and its squared volatility. In the above-mentioned papers, the authors assume constant interest rates. Although the assumption of constant interest rates leads to highly tractable FX models, empirical results have confirmed that such models do not reflect the market reality, especially for long-dated hybrid foreign exchange and interest rate products, such as PRDCs (Power Reverse Dual Currency notes and bonds) or FX-TARNs (Forex Target Redemption Notes). For these products, the fluctuations of both the exchange rate and the interest rates are critical, and thus the postulate of constant interest rates in both economies is clearly inappropriate for reliable valuation and hedging as was pointed out, for instance, in Chapter 11 of the monograph by Clark (2011). In Ahlip and Rutkowski (2015), we proposed a hybrid model, which combines methods from the foreign exchange and fixed income markets. This model is extended here to include double exponential jumps in the squared volatility of the exchange rate, but still with continuous processes for the domestic and foreign interest rates. The double exponential jump diffusion model, on the other hand, via the introduction of asymmetric jumps, is capable of characterizing distributions which are highly skewed and leptokurtic. As shown by Carr and Wu (2004) and Kou and Wang (2004) options valuation under the double exponential jump diffusion model leads to the analytic tractability of pricing results, and facilitates efficient computation of the hedging parameters.

Let us comment briefly on the existing literature in the same vein. Van Haastrecht *et al.* (2009) have extended the stochastic volatility model of Schöbel and Zhu (1999) to equity/currency derivatives by including stochastic interest rates and assuming all driving model factors to be instantaneously correlated. Since their model is based on Gaussian processes, it enjoys analytical tractability even in the most general case of a full correlation structure. By contrast, when the squared volatility is driven by the CIR process and the interest rate is driven either by the Vasicek (1977) or the

Cox *et al.* (1985) process, a full correlation structure leads to intractability of equity options even under a partial correlation of the driving factors. This has been documented by, among others, Van Haastrecht and Pelsser (2011) and Grzelak and Oosterlee (2011, 2012) who examined, in particular, the Heston/Vasicek and Heston/CIR hybrid models (see also Grzelak *et al.* (2012), where the Schöbel–Zhu/Hull–White and Heston/Hull–White models for equity derivatives are studied). Kuo *et al.* derive analytical approximations for finite-horizon American options by considering a double exponential jump diffusion model. Ramezani *et al.* (2007) provide an empirical assessment of the double exponential jump diffusion model using a maximum likelihood estimation procedure. More recently Mi-Hsiu Chiang *et al.* (2016) consider pricing currency options under double exponential jump model, where the interest-rate regime shifts are driven by hidden Markov chains. The authors have shown that the skewness and Kurtosis, for the 100JPY/USD and the EUR/USD spot-FX rates under the double exponential jump diffusion model most closely matches the sample data.

Our goal is to derive a semi-analytical solution for prices of plain-vanilla currency options in a model in which the volatility component is specified by the extended Heston model with double exponential jumps, whereas the short-term interest rates for the domestic and foreign economies are governed by the independent CIR processes. The model thus incorporates important empirical characteristics of exchange rate return variability: (a) the correlation between the exchange rate and its stochastic volatility, (b) the presence of jumps in the exchange rate and volatility processes and (c) the random character of interest rates. The practical importance of this feature of newly developed FX models is rather clear in view of the existence of complex FX products that have a long lifetime and are sensitive to smiles or skews in the market.

The paper is organised as follows. In Section 2, we set the foreign exchange model examined in this work. The options pricing problem is introduced in Section 3. The main result, Theorem 4.1 of Section 4, furnishes the pricing formula for currency options. It is worth stressing that the independence of volatility and interest rates appears to be a crucial assumption from the point of view of analytical tractability and thus it cannot be relaxed. Numerical illustrations of our method are provided in Section 5 where the diffusion and jump-diffusion models are compared.

2 The Heston-Double Exponential/CIR Foreign Exchange Model

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be an underlying probability space. Let the exchange rate $Q = (Q_t)_{t\in[0,T]}$, its instantaneous squared volatility $v = (v_t)_{t\in[0,T]}$, the domestic short-term interest rate $r = (r_t)_{t\in[0,T]}$, and the foreign short-term interest rate $\widehat{r} = (\widehat{r}_t)_{t\in[0,T]}$ be governed by the following system of SDEs:

$$\begin{cases} dQ_t = Q_t\big(r_t - \widehat{r}_t\big)dt + Q_t\,\sqrt{v_t}\,dW_t^Q + Q_{t-}\,dZ_t^Q, \\ dv_t = \big(\theta - \kappa v_t\big)dt + \sigma_v\sqrt{v_t}\,dW_t^v + dZ_t^v, \\ dr_t = \big(a_d - b_d r_t\big)dt + \sigma_d\sqrt{r_t}\,dW_t^d, \\ d\widehat{r}_t = \big(a_f - b_f\widehat{r}_t\big)dt + \sigma_f\sqrt{\widehat{r}_t}\,dW_t^f. \end{cases} \tag{1}$$

We work under the following standing assumptions:

(A.1) Processes $W^Q = (W_t^Q)_{t\in[0,T]}$, $W^v = (W_t^v)_{t\in[0,T]}$ are correlated Brownian motions with a constant correlation coefficient, so that the quadratic covariation between the processes W^Q and W^v satisfies $d[W^Q, W^v]_t = \rho\,dt$ for some constant $\rho \in [-1, 1]$.
(A.2) Processes $W^d = (W_t^d)_{t\in[0,T]}$ and $W^f = (W_t^f)_{t\in[0,T]}$ are independent Brownian motions and are also independent of the Brownian motions W^Q and W^v; hence the processes Q, r and $\widehat{r}$ are independent.
(A.3) The process $Z_t^Q = \sum_{k=1}^{N_t^Q} J_k^Q$ is the compound Poisson process; specifically, the Poisson process N^Q has the intensity $\lambda_Q > 0$ and the random variables $\ln(1 + J_k^Q)$, $k = 1, 2, \dots$ have the probability distribution $N(\ln[1 + \mu_Q] - \frac{1}{2}\sigma_Q^2, \sigma_Q^2)$; hence the jump sizes $(J_k^Q)_{k=1}^{\infty}$ are lognormally distributed on $(-1, \infty)$ with mean $\mu_Q > -1$.

(A.4) The process $Z_t^v = \sum_{k=1}^{N_t^v} J_k^v$ is the compound Poisson process; specifically, the Poisson process N^v has the intensity $\lambda_v > 0$ and the jump sizes J_k^v are independent identically distributed (i.i.d) nonnegative random variables such that $Y = \log J$, has an asymmetric double exponential distribution with density

$$\nu_Y(y) = p.\eta_1 e^{-\eta_1 y}\mathbb{1}_{y\geq 0} + q.\eta_2 e^{\eta_2 y}\mathbb{1}_{y<0}, \quad \eta_1 > 1, \quad \eta_2 > 0, \tag{2}$$

where $p, q \geq 0, p + q = 1$.
(A.5) The Poisson process N^v and sequence of random variables $(J_k^v)_{k=1}^{\infty}$ are independent of the Brownian motions W^Q, W^v, W^d, W^f.
(A.6) The model's parameters satisfy the stability conditions: $2\theta > \sigma_v^2 > 0$, $2a_d > \sigma_d^2 > 0$ and $2a_f > \sigma_f^2 > 0$ (see, for instance, Wong and Heyde (2004)).

Note that we postulate that the instantaneous squared volatility process v, the domestic short-term interest rate r, and the foreign interest rate $\widehat{r}$ are independent stochastic processes. We will argue in what follows that this assumption is indeed crucial for analytical tractability. For brevity, we refer to the foreign exchange model given by SDEs (1) under Assumptions (A.1)–(A.6) as the Heston/CIR jump-diffusion FX model.

3 Foreign Exchange Call Option

We will first establish the general representation for the value of the foreign exchange (i.e., currency) European call option with maturity $T > 0$ and a constant strike level $K > 0$. The probability measure $\mathbb{P}$ is interpreted as the domestic spot martingale measure (i.e., the domestic risk-neutral probability). We denote by $\mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]}$ the filtration generated by the Brownian motions W^Q, W^v, W^d, W^f and the compound Poisson process Z^v. We write $\mathbb{E}_t^{\mathbb{P}}(\cdot)$ and $\mathbb{P}_t(\cdot)$ to denote the conditional expectation and the conditional probability under $\mathbb{P}$ with respect to the σ-field $\mathcal{F}_t$, respectively. In our computations, we will adopt the 'domestic' point of view, which will frequently be represented by the subscript d. Similarly, we will use the subscript f when referring to a foreign denominated variable. Hence the arbitrage price $C_t(T,K)$ of the foreign exchange call option at time $t \in [0,T]$ is given as the conditional expectation with respect to the σ-field $\mathcal{F}_t$ of the option's payoff at expiration, discounted by the domestic money market account, that is,

$$C_t(T,K) = \mathbb{E}_t^{\mathbb{P}}\left\{\exp\left(-\int_t^T r_u\,du\right)C_T(T,K)\right\} = \mathbb{E}_t^{\mathbb{P}}\left\{\exp\left(-\int_t^T r_u\,du\right)(Q_T - K)^+\right\}$$

or, equivalently,

$$C_t(T,K) = \mathbb{E}_t^{\mathbb{P}}\left\{\exp\left(-\int_t^T r_u\,du\right)Q_T\mathbb{1}_{\{Q_T>K\}}\right\} - K\mathbb{E}_t^{\mathbb{P}}\left\{\exp\left(-\int_t^T r_u\,du\right)\mathbb{1}_{\{Q_T>K\}}\right\}.$$

Similarly, the arbitrage price of the domestic discount bond maturing at time T equals, for every $t \in [0,T]$,

$$B_d(t,T) = \mathbb{E}_t^{\mathbb{P}}\left\{\exp\left(-\int_t^T r_u\,du\right)\right\}$$

and an analogous formula holds for the price process $B_f(t,T)$ of the foreign discount bond under the foreign spot martingale measure (see Musiela and Rutkowski (2005, Chapter 14)).

As a preliminary step towards the general valuation result presented in Section 4, we state the following well-known proposition (see, e.g, Cox *et al.* (1985) or Chapter 10 in Musiela and Rutkowski (2005)). It is worth stressing that we use here, in particular, the postulated independence of the foreign interest rate $\widehat{r}$ and the exchange rate process Q. Under this standing assumption, the dynamics of the foreign bond price $B_f(t,T)$ under the domestic spot martingale measure $\mathbb{P}$ can be seen as an immediate consequence of formula (14.3) in Musiela and Rutkowski (2005). The simple form of the dynamics of $B_f(t,T)$ under $\mathbb{P}$ is a consequence of the postulated independence of W^f and W^Q (see Assumption (A.2)). This crucial feature underpins our further calculations and thus it cannot be easily relaxed.

Proposition 3.1. *The prices at date t of the domestic and foreign discount bonds maturing at time $T > t$ in the CIR model are given by the following expressions*

$$\begin{aligned} B_d(t,T) &= \exp\big(m_d(t,T) - n_d(t,T)r_t\big), \\ B_f(t,T) &= \exp\big(m_f(t,T) - n_f(t,T)\widehat{r}_t\big), \end{aligned}$$

where for $i \in \{d, f\}$

$$\begin{aligned} m_i(t,T) &= \frac{2a_i}{\sigma_i^2} \log\left[\frac{\gamma_i\, e^{\frac{1}{2}b_i(T-t)}}{\gamma_i \cosh(\gamma_i(T-t)) + \frac{1}{2}\, b_i \sinh(\gamma_i(T-t))}\right], \\ n_i(t,T) &= \frac{\sinh(\gamma_i(T-t))}{\gamma_i \cosh(\gamma_i(T-t)) + \frac{1}{2}\, b_i \sinh(\gamma_i(T-t))}, \end{aligned}$$

and

$$\gamma_i = \frac{1}{2}\sqrt{b_i^2 + 2\sigma_i^2}.$$

The dynamics of the domestic and foreign bond prices under the domestic spot martingale measure $\mathbb{P}$ are given by

$$\begin{aligned} dB_d(t,T) &= B_d(t,T)\big(r_t\, dt - \sigma_d n_d(t,T)\sqrt{r_t}\, dW_t^d\big), \\ dB_f(t,T) &= B_f(t,T)\big(\widehat{r}_t\, dt - \sigma_f n_f(t,T)\sqrt{\widehat{r}_t}\, dW_t^f\big). \end{aligned}$$

The following result is also well known (see, for instance, Section 14.1.1 in Musiela and Rutkowski (2005)).

Lemma 3.1. *The forward exchange rate $F(t,T)$ at time t for settlement date T equals*

$$F(t,T) = \frac{B_f(t,T)}{B_d(t,T)}\, Q_t. \tag{3}$$

Since manifestly $Q_T = F(T,T)$, the option's payoff at expiration can also be expressed as follows

$$C_T(T,K) = F(T,T)\mathbb{1}_{\{F(T,T)>K\}} - K\mathbb{1}_{\{F(T,T)>K\}}.$$

Consequently, the option's value at time $t \in [0,T]$ admits the following representation

$$\begin{aligned} C_t(T,K) = {} & \mathbb{E}_t^{\mathbb{P}}\left\{\exp\left(-\int_t^T r_u\, du\right) F(T,T)\mathbb{1}_{\{F(T,T)>K\}}\right\} \\ & - K\mathbb{E}_t^{\mathbb{P}}\left\{\exp\left(-\int_t^T r_u\, du\right)\mathbb{1}_{\{F(T,T)>K\}}\right\}. \end{aligned}$$

In what follows, we will frequently use the notation $x_t = \ln F(t,T)$ where $t \in [0,T]$.

4 Pricing Formula for the Currency Call Option

We are in a position to state the main result of the paper, which furnishes a semi-analytical formula for the arbitrage price of the currency call option of European style under the Heston-Double Exponential/CIR Jump Diffusion Model (2).

Remark 4.1. *Since the proof of Theorem 4.1 relies on the derivation of the conditional characteristic function of the logarithm of the exchange rate, any suitable version of the Fourier inversion technique or simulation technique can be applied to obtain the option price. For a detailed analysis of these methods, the interested reader is referred to, for instance, Carr and Madan (1999, 2009) or Lord and Kahl (2007, 2010) and the references therein, as well as the recent papers by Bernard et al. (2012) and Levendorskiĭ (2012) who developed and examined in detail methods with essential improvements in accuracy and/or efficiency.*

Theorem 4.1. *Let the foreign exchange model be given by SDEs (1) under Assumptions (A.1)–(A.6). Then the price of the European currency call option equals, for every $t \in [0,T]$,*

$$C_t(T,K) = Q_t B_f(t,T) P_1\big(t, Q_t, v_t, r_t, \widehat{r}_t, K\big) - K B_d(t,T) P_2\big(t, Q_t, v_t, r_t, \widehat{r}_t, K\big)$$

where the bond prices $B_d(t,T)$ and $B_f(t,T)$ are given in Proposition 3.1, and the functions P_1 and P_2 are given by, for $j = 1, 2$,

$$P_j\big(t, Q_t, v_t, r_t, \widehat{r}_t, K\big) = \frac{1}{2} + \frac{1}{\pi} \int_0^\infty \mathrm{Re}\left(f_j(\phi)\, \frac{\exp(-i\phi \ln K)}{i\phi} \right) d\phi \tag{4}$$

where the $\mathcal{F}_t$-conditional characteristic functions $f_j(\phi) = f_j(\phi, t, Q_t, v_t, r_t, \widehat{r}_t)$, $j = 1, 2$ of the random variable $x_T = \ln(Q_T)$ under the probability measure $\widehat{\mathbb{P}}_T$ (see Definition 4.2) and $\mathbb{P}_T$ (see Definition 4.1), respectively, are given by

$$\begin{aligned}
f_1(\phi) \;=\;& c_t \exp\left[\lambda_Q \tau \left((1+\mu_Q)^{i\phi} e^{-\frac{1}{2}(\phi^2 + i\phi)\sigma_Q^2} - 1 \right)\right] \\
&\times \exp\left[-\left(i\phi \lambda_Q \mu_Q + \frac{(1+i\phi)\rho}{\sigma_v}(v_t + \theta\tau) \right)\right] \\
&\times \exp\left[\lambda_v \sigma_v \tau \left(\frac{p\eta_1}{\sigma_v \eta_1 + (1+i\phi)\rho} + \frac{q\eta_2}{\sigma_v \eta_2 - (1+i\phi)\rho} - 1 \right)\right] \\
&\times \exp\left[- i\phi \left(n_d(t,T) r_t + \int_t^T a_d n_d(u,T)\, du \right)\right] \\
&\times \exp\left[(1+i\phi) \left(n_f(t,T) \widehat{r}_t + \int_t^T a_f n_f(u,T)\, du \right)\right] \\
&\times \exp\left[- G_1(\tau, s_1, s_2) v_t - G_2(\tau, s_3, s_4) r_t - G_3(\tau, s_5, s_6) \widehat{r}_t \right] \\
&\times \exp\left[- \theta H_1(\tau, s_1, s_2) - a_d H_2(\tau, s_3, s_4) - a_f H_3(\tau, s_5, s_6) \right]
\end{aligned} \tag{5}$$

and

$$\begin{aligned}
f_2(\phi) \;=\;& c_t \exp\left[\lambda_Q \tau \left((1+\mu_Q)^{i\phi} e^{-\frac{1}{2}(\phi^2 + i\phi)\sigma_Q^2} - 1 \right)\right] \\
&\times \exp\left[-\left(i\phi \lambda_Q \mu_Q \tau + \frac{i\phi\rho}{\sigma_v}(v_t + \theta\tau) \right)\right] \\
&\times \exp\left[\lambda_v \sigma_v \tau \left(\frac{p\eta_1}{\sigma_v \eta_1 + i\phi\rho} + \frac{q\eta_2}{\sigma_v \eta_2 - i\phi\rho} - 1 \right)\right] \\
&\times \exp\left[(1 - i\phi) \left(n_d(t,T) r_t + \int_t^T a_d n_d(u,T)\, du \right)\right] \\
&\times \exp\left[i\phi \left(n_f(t,T) \widehat{r}_t + \int_t^T a_f n_f(u,T)\, du \right)\right] \\
&\times \exp\left[- G_1(\tau, q_1, q_2) v_t - G_2(\tau, q_3, q_4) r_t - G_3(\tau, q_5, q_6) \right] \\
&\times \exp\left[- \theta H_1(\tau, q_1, q_2) - a_d H_2(\tau, q_3, q_4) - a_f H_3(\tau, q_5, q_6) \right]
\end{aligned} \tag{6}$$

where the functions $G_1, G_2, G_3, H_1, H_2, H_3$ are given in Lemma 4.2 and c_t equals

$$c_t = \exp\big(i\phi x_t\big) = \exp\left(i\phi \ln F(t,T)\right).$$

Moreover, the constants $s_1, s_2, s_3, s_4, s_5, s_6$ are given by

$$\begin{aligned} s_1 &= -\frac{(1+i\phi)\rho}{\sigma_v}, \\ s_2 &= -\frac{(1+i\phi)^2(1-\rho^2)}{2} - \frac{(1+i\phi)\rho\kappa}{\sigma_v} + \frac{1+i\phi}{2}, \\ s_3 &= 0,\ \ s_4 = -i\phi,\ \ s_5 = 0,\ \ s_6 = 1+i\phi, \end{aligned} \tag{7}$$

and the constants $q_1, q_2, q_3, q_4, q_5, q_6$ equal

$$\begin{aligned} q_1 &= -\frac{i\phi\rho}{\sigma_v}, \\ q_2 &= -\frac{(i\phi)^2(1-\rho^2)}{2} - \frac{i\phi\rho\kappa}{\sigma_v} + \frac{i\phi}{2}, \\ q_3 &= 0,\ \ q_4 = 1-i\phi,\ \ q_5 = 0\ q_6 = i\phi. \end{aligned} \tag{8}$$

4.1 Auxiliary Results

The proof of Theorem 4.1 hinges on a number of lemmas. We start by stating the well known result, which can be easily obtained from Proposition 8.6.3.4 in Jeanblanc *et al.* (2009).

Let us denote $\tau = T - t$ and let us set, for all $0 \le t < T$, and $\nu_Z(\cdot)$ the double exponential distribution given by (2) in assumption (A3).

$$J^Q(t,T) := \sum_{k=N_t^Q+1}^{N_T^Q} \ln(1+J_k^Q). \tag{9}$$

Note that we use here Assumptions (A.3)–(A.5). The property (A.3) (resp. (A.4)) implies that the random variable $J^Q(t,T)$ (resp. $Z_T^v - Z_t^v$) is independent of the σ-field $\mathcal{F}_t$. Let ν_1 stand for the Gaussian distribution $N\big(\ln(1+\mu_Q) - \frac{1}{2}\sigma_Q^2, \sigma_Q^2\big)$ and let ν_Z stand for the double exponential distribution (2) in assumption (A4).

Lemma 4.1. *(i) Under Assumptions (A.3) and (A.5), the following equalities are valid*

$$\begin{aligned} \mathbb{E}_t^{\mathbb{P}}\Big\{\exp\big(i\phi J^Q(t,T)\big)\Big\} &= \mathbb{E}_t^{\mathbb{P}}\Bigg\{\exp\Bigg(i\phi \sum_{k=N_t^Q+1}^{N_T^Q} \ln\big(1+J_k^Q\big)\Bigg)\Bigg\} \\ &= \exp\left[\lambda_Q \tau \int_{-\infty}^{+\infty} \left(e^{i\phi z} - 1\right)\nu_1(dz)\right] \\ &= \exp\left[\lambda_Q \tau \left((1+\mu_Q)^{i\phi} e^{-\frac{1}{2}\sigma_Q^2(\phi^2+i\phi)} - 1\right)\right]. \end{aligned}$$

Under Assumptions (A.3) and (A.4), the following equalities are valid for $c = a + bi$ with $-\eta_2 \le a \le \eta_1$

$$\begin{aligned} \mathbb{E}_t^{\mathbb{P}}\Big\{\exp\big(c\,(Z_T^v - Z_t^v)\big)\Big\} &= \mathbb{E}_t^{\mathbb{P}}\Bigg\{\exp\Bigg(c \sum_{k=N_t^v+1}^{N_T^v} J_k^v\Bigg)\Bigg\} \\ &= \exp\left[\lambda_v \tau \int_{-\infty}^{\infty} (e^{cz} - 1)\,\nu_Z(dz)\right] \\ &= \exp\left[\lambda_v \tau \left(\frac{p\eta_1}{\eta_1 - c} + \frac{q\eta_2}{\eta_2 + c} - 1\right)\right]. \end{aligned}$$

The next result extends Lemma 6.1 in Ahlip and Rutkowski (2009) (see also Duffie *et al.* (2000)) where the model without the jump component in the dynamics of v was examined.

Lemma 4.2. *Let the dynamics of processes v, r and $\widehat{r}$ be given by SDEs (1) with independent Brownian motions W^v, W^d and W^f. For any complex numbers μ, λ, $\widetilde{\mu}$, $\widetilde{\lambda}$, $\widehat{\mu}$, $\widehat{\lambda}$, we set*

$$F(\tau, v_t, r_t, \widehat{r}_t) = \mathbb{E}_t^{\mathbb{P}}\left\{ \exp\left(-\lambda v_T - \mu \int_t^T v_u\,du - \widetilde{\lambda} r_T - \widetilde{\mu}\int_t^T r_u\,du - \widehat{\lambda}\widehat{r}_T - \widehat{\mu}\int_t^T \widehat{r}_u\,du \right)\right\}.$$

Then

$$\begin{aligned} F(\tau, v_t, r_t, \widehat{r}_t) = \exp\big[&- G_1(\tau,\lambda,\mu)v_t - G_2(\tau,\widetilde{\lambda},\widetilde{\mu})r_t - G_3(\tau,\widehat{\lambda},\widehat{\mu})\widehat{r}_t \\ &- \theta H_1(\tau,\lambda,\mu) - a_d H_2(\tau,\widetilde{\lambda},\widetilde{\mu}) - a_f H_3(\tau,\widehat{\lambda},\widehat{\mu})\big] \end{aligned}$$

where

$$\begin{aligned} G_1(\tau,\lambda,\mu) &= \frac{\lambda[(\gamma+\kappa) + e^{\gamma\tau}(\gamma-\kappa)] + 2\mu(e^{\gamma\tau}-1)}{\sigma_v^2\lambda\,(e^{\gamma\tau}-1) + \gamma - \kappa + e^{\gamma\tau}\,(\gamma+\kappa)}, \\ G_2(\tau,\widetilde{\lambda},\widetilde{\mu}) &= \frac{\widetilde{\lambda}[(\widetilde{\gamma}+b_d) + e^{\widetilde{\gamma}\tau}(\widetilde{\gamma}-b_d)] + 2\widetilde{\mu}(e^{\widetilde{\gamma}\tau}-1)}{\sigma_d^2\widetilde{\lambda}\,(e^{\widetilde{\gamma}\tau}-1) + \widetilde{\gamma} - b_d + e^{\widetilde{\gamma}\tau}\,(\widetilde{\gamma}+b_d)}, \\ G_3(\tau,\widehat{\lambda},\widehat{\mu}) &= \frac{\widehat{\lambda}[(\widehat{\gamma}+b_f) + e^{\widehat{\gamma}\tau}(\widehat{\gamma}-b_f)] + 2\widehat{\mu}(e^{\widehat{\gamma}\tau}-1)}{\sigma_f^2\widehat{\lambda}\,(e^{\widehat{\gamma}\tau}-1) + \widehat{\gamma} - b_f + e^{\widehat{\gamma}\tau}\,(\widehat{\gamma}+b_f)}, \end{aligned}$$

and

$$\begin{aligned} H_1(\tau,\lambda,\mu) &= \int_0^\tau \left(G_1(t,\lambda,\mu) + \frac{\lambda_v}{\theta}\left(\frac{pG_1}{\eta_1+G_1} + \frac{q\eta_2}{\eta_2-G_1}\right)\right)dt, \\ H_2(\tau,\widetilde{\lambda},\widetilde{\mu}) &= -\frac{2}{\sigma_d^2}\ln\left(\frac{2\widetilde{\gamma}e^{\frac{(\widetilde{\gamma}+b_d)\tau}{2}}}{\sigma_d^2\widetilde{\lambda}\,(e^{\widetilde{\gamma}\tau}-1) + \widetilde{\gamma} - b_d + e^{\widetilde{\gamma}\tau}\,(\widetilde{\gamma}+b_d)}\right), \\ H_3(\tau,\widehat{\lambda},\widehat{\mu}) &= -\frac{2}{\sigma_f^2}\ln\left(\frac{2\widehat{\gamma}e^{\frac{(\widehat{\gamma}+b_f)\tau}{2}}}{\sigma_f^2\widehat{\lambda}\,(e^{\widehat{\gamma}\tau}-1) + \widehat{\gamma} - b_f + e^{\widehat{\gamma}\tau}\,(\widehat{\gamma}+b_f)}\right), \end{aligned}$$

where we denote $\gamma = \sqrt{\kappa^2 + 2\sigma_v^2\mu}$, $\widetilde{\gamma} = \sqrt{b_d^2 + 2\sigma_d^2\widetilde{\mu}}$ and $\widehat{\gamma} = \sqrt{b_f^2 + 2\sigma_f^2\widehat{\mu}}$.

Proof. For the reader's convenience, we sketch the proof of the lemma. Let us set, for $t \in [0,T]$,

$$M_t = F(\tau, v_t, r_t, \widehat{r}_t)\exp\left(-\mu\int_0^t v_u\,du - \widetilde{\mu}\int_0^t r_u\,du - \widehat{\mu}\int_0^t \widehat{r}_u\,du\right). \tag{10}$$

Then the process $M = (M_t)_{t\in[0,T]}$ satisfies

$$M_t = \mathbb{E}_t^{\mathbb{P}}\left\{ \exp\left(-\lambda v_T - \mu\int_0^T v_u\,du - \widetilde{\lambda} r_T - \widetilde{\mu}\int_0^T r_u\,du - \widehat{\lambda}\widehat{r}_T - \widehat{\mu}\int_0^T r_u\,du\right)\right\}$$

and thus it is an $\mathbb{F}$-martingale under $\mathbb{P}$. By applying the Itô formula to the right-hand side in (10) and by setting the drift term in the dynamics of M to be zero, we deduce that the function $F(\tau, v, r, \widehat{r})$ satisfies the following partial integro-differential equation (PIDE)

$$\begin{aligned} &-\frac{\partial F}{\partial\tau} + \frac{1}{2}\sigma_v^2 v\,\frac{\partial^2 F}{\partial v^2} + \lambda_v\int_{-\infty}^{\infty}\big(F(\tau, v+z, r, \widehat{r}) - F(\tau, v, r, \widehat{r})\big)\nu_2(dz) \\ &+\frac{1}{2}\sigma_d^2 r\,\frac{\partial^2 F}{\partial r^2} + \frac{1}{2}\sigma_f^2\widehat{r}\,\frac{\partial^2 F}{\partial\widehat{r}^2} + (\theta - \kappa v)\,\frac{\partial F}{\partial v} + (a_d - b_d r)\,\frac{\partial F}{\partial r} \\ &\qquad + (a_f - b_f\widehat{r})\,\frac{\partial F}{\partial\widehat{r}} - (\mu v + \widetilde{\mu} r + \widehat{\mu}\widehat{r})F = 0 \end{aligned}$$

with the initial condition $F(0,v,r,\widehat{r}) = \exp(-\lambda v - \widetilde{\lambda} r - \widehat{\lambda}\widehat{r})$. We search for a solution to this PIDE in the form

$$\begin{aligned} F(\tau,v,r,\widehat{r}) = \exp\big[& - G_1(\tau,\lambda,\mu)v - G_2(\tau,\widetilde{\lambda},\widetilde{\mu})r - G_3(\tau,\widehat{\lambda},\widehat{\mu})\widehat{r} \\ & - \theta H_1(\tau,\lambda,\mu) - a_d H_2(\tau,\widetilde{\lambda},\widetilde{\mu}) - a_f H_3(\tau,\widehat{\lambda},\widehat{\mu})\big] \end{aligned}$$

with

$$G_1(0,\lambda,\mu) = \lambda, \quad G_2(0,\widetilde{\lambda},\widetilde{\mu}) = \widetilde{\lambda}, \quad G_3(0,\widehat{\lambda},\widehat{\mu}) = \widehat{\lambda},$$

and

$$H_1(0,\lambda,\mu) = H_2(0,\widetilde{\lambda},\widetilde{\mu}) = H_3(0,\widehat{\lambda},\widehat{\mu}) = 0.$$

By substituting this expression in the PIDE and using part (ii) in Lemma 4.1, we obtain the following system of ODEs for the functions $G_1, G_2, G_3, H_1, H_2, H_3$ (for brevity, we suppress the last three arguments)

$$\begin{aligned} \frac{\partial G_1(\tau)}{\partial \tau} &= -\frac{1}{2}\sigma_v^2 G_1^2(\tau) - \kappa G_1(\tau) + \mu, \\ \frac{\partial H_1(\tau)}{\partial \tau} &= G_1(\tau) + \frac{\lambda_v}{\theta}\left(\frac{p\eta_1}{\eta_1 + G_1(\tau)} + \frac{q\eta_2}{\eta_2 - G_1(\tau)} - 1\right) \\ \frac{\partial G_2(\tau)}{\partial \tau} &= -\frac{1}{2}\sigma_d^2 G_2^2(\tau) - b_d G_2(\tau) + \widetilde{\mu}, \\ \frac{\partial H_2(\tau)}{\partial \tau} &= G_2(\tau), \\ \frac{\partial G_3(\tau)}{\partial \tau} &= -\frac{1}{2}\sigma_f^2 G_3^2(\tau) - b_f G_3(\tau) + \widehat{\mu}, \\ \frac{\partial H_3(\tau)}{\partial \tau} &= G_3(\tau). \end{aligned}$$

By solving these equations, we obtain the stated formulae. □

Under the assumptions of Lemma 4.2, it is possible to factorise F as a product of two conditional expectations. This means that the functions G_1 (H_1), G_2 (H_2), and G_3 (H_3) are of the same form, except that they correspond to different sets of parameters, θ, κ, σ_v for G_1, H_1, a_d, b_d, σ_r for G_2, H_2 and a_f, b_f, σ_f for G_3, H_3. Note, however, that the roles played by the processes v, r and $\widehat{r}$ in our model are clearly different.

It should also be stressed that no closed-form analytical expression for $F(\tau, v_t, r_t, \widehat{r}_t)$ is available in the case of correlated Brownian motions W^v, W^r, W^f. Brigo and Alfonsi (2005), who dealt with a similar issue in a different context, proposed to use a simple Gaussian approximation, instead of searching for an exact solution. More recently, Grzelak and Oosterlee (2011) proposed more sophisticated approximations in the framework of the Heston/CIR hybrid model. We do not follow this line of research here and we focus instead on finding a semi-analytical solution, since this goal can be achieved under Assumptions (A.1)–(A.6).

Let us now introduce a convenient change of the underlying probability measure, from the domestic spot martingale measure $\mathbb{P}$ to the domestic forward martingale measure $\mathbb{P}_T$.

Definition 4.1. The *domestic forward martingale measure* $\mathbb{P}_T$, equivalent to $\mathbb{P}$ on $(\Omega, \mathcal{F}_T)$, is defined by the Radon-Nikodým derivative process $\eta = (\eta_t)_{t\in[0,T]}$ where

$$\eta_t = \frac{d\mathbb{P}_T}{d\mathbb{P}}\Big|_{\mathcal{F}_t} = \exp\Big(- \int_0^t \sigma_d n_d(u,T)\sqrt{r_u}\, dW_u^d - \frac{1}{2}\int_0^t \sigma_d^2 n_d^2(u,T) r_u\, du\Big). \tag{11}$$

An application of Girsanov's theorem shows that the process $W^T = (W_t^T)_{t\in[0,T]}$, which is given by the equality

$$W_t^T = W_t^d + \int_0^t \sigma_d n_d(u,T)\sqrt{r_u}\, du, \tag{12}$$

is the Brownian motion under the domestic forward martingale measure $\mathbb{P}_T$. Using the standard change of a numéraire technique, one can check that the price of the European foreign exchange call option admits the following representation under the probability measure $\mathbb{P}_T$

$$C_t(T,K) = B_d(t,T)\,\mathbb{E}_t^{\mathbb{P}_T}\big(F(T,T)\mathbb{1}_{\{F(T,T)>K\}}\big) - KB_d(t,T)\,\mathbb{E}_t^{\mathbb{P}_T}\big(\mathbb{1}_{\{F(T,T)>K\}}\big). \tag{13}$$

The following auxiliary result is easy to establish and thus its proof is omitted. Recall that $J^Q(t,T)$ is given by equality (9).

Lemma 4.3. *Under Assumptions (A.1)–(A.6), the dynamics of the forward exchange rate $F(t,T)$ under the domestic forward martingale measure $\mathbb{P}_T$ are given by the SDE*

$$dF(t,T) = F(t,T)\,\Big(dZ_t^Q - \lambda_Q\mu_Q dt + \sqrt{v_t}\,dW_t^Q + \sigma_d\, n_d(t,T)\sqrt{r_t}\,dW_t^T - \sigma_f n_f(t,T)\sqrt{\widehat{r}_t}\,dW_t^f\Big)$$

or, equivalently,

$$F(T,T) = F(t,T)\exp\left(J^Q(t,T) - \lambda_Q\mu_Q(T-t) + \int_t^T \widetilde{\sigma}_F(u,T)\cdot d\widetilde{W}_u^T - \frac{1}{2}\int_t^T \|\widetilde{\sigma}_F(u,T)\|^2\,du\right)$$

where the dot $\cdot$ denotes the inner product in $\mathbb{R}^3$, $(\widetilde{\sigma}_F(t,T))_{t\in[0,T]}$ is the $\mathbb{R}^3$-valued process (row vector) given by

$$\widetilde{\sigma}_F(t,T) = \big[\sqrt{v_t},\ \sigma_d n_d(t,T)\sqrt{r_t},\ -\sigma_f n_f(t,T)\sqrt{\widehat{r}_t}\,\big]$$

and $\widetilde{W}^T = (\widetilde{W}_t^T)_{t\in[0,T]}$ is the $\mathbb{R}^3$-valued process (column vector) given by $\widetilde{W}^T = \big[W^Q,\, W^T,\, W^f\big]^*$.

It is easy to check that, under Assumptions (A.1)–(A.6), the process $\widetilde{W}^T$ is the three-dimensional standard Brownian motion under $\mathbb{P}_T$. In view of Lemma 4.3, we have that

$$\begin{aligned}
&B_d(t,T)\,\mathbb{E}_t^{\mathbb{P}_T}\big(F(T,T)\mathbb{1}_{\{F(T,T)>K\}}\big)\\
&\quad= B_d(t,T)\,\mathbb{E}_t^{\mathbb{P}_T}\Bigg\{F(t,T)\exp\bigg(J^Q(t,T) - \lambda_Q\mu_Q(T-t) + \int_t^T \widetilde{\sigma}_F(u,T)\cdot d\widetilde{W}_u^T\\
&\qquad\qquad -\frac{1}{2}\int_t^T \|\widetilde{\sigma}_F(u,T)\|^2\,du\bigg)\,\mathbb{1}_{\{F(T,T)>K\}}\Bigg\}\\
&\quad= Q_t B_f(t,T)\,\mathbb{E}_t^{\mathbb{P}_T}\Bigg\{\exp\bigg(J^Q(t,T) - \lambda_Q\mu_Q(T-t) + \int_t^T \widetilde{\sigma}_F(u,T)\cdot d\widetilde{W}_u^T\\
&\qquad\qquad -\frac{1}{2}\int_t^T \|\widetilde{\sigma}_F(u,T)\|^2\,du\bigg)\,\mathbb{1}_{\{F(T,T)>K\}}\Bigg\}.
\end{aligned}$$

To deal with the first term in the right-hand side of (13), we introduce another auxiliary probability measure.

Definition 4.2. The *modified domestic forward martingale measure* $\widehat{\mathbb{P}}_T$, equivalent to $\mathbb{P}_T$ on $(\Omega,\mathcal{F}_T)$, is defined by the Radon-Nikodým derivative process $\widehat{\eta} = (\widehat{\eta}_t)_{t\in[0,T]}$ where

$$\widehat{\eta}_t = \frac{d\widehat{\mathbb{P}}_T}{d\mathbb{P}_T}\Big|_{\mathcal{F}_t} = \exp\left(\int_0^t \widetilde{\sigma}_F(u,T)\cdot d\widetilde{W}_u^T - \frac{1}{2}\int_0^t \|\widetilde{\sigma}_F(u,T)\|^2\,du\right).$$

Using Lemma 4.3 and equation (3), we obtain

$$B_d(t,T)\,\mathbb{E}_t^{\mathbb{P}_T}\big(F(T,T)\mathbb{1}_{\{F(T,T)>K\}}\big) = Q_t B_f(t,T)\,\frac{\mathbb{E}_t^{\mathbb{P}_T}\big(\mathbb{1}_{\{F(T,T)>K\}}\widehat{\eta}_T\big)}{\mathbb{E}_t^{\mathbb{P}_T}(\widehat{\eta}_T)}$$

and thus the Bayes formula and Definition 4.2 yield

$$B_d(t,T)\,\mathbb{E}_t^{\mathbb{P}_T}\big(F(T,T)\mathbb{1}_{\{F(T,T)>K\}}\big) = Q_t B_f(t,T)\,\mathbb{E}_t^{\widehat{\mathbb{P}}_T}\big(\mathbb{1}_{\{F(T,T)>K\}}\big).$$

This shows that $\widehat{\mathbb{P}}_T$ is a martingale measure associated with the choice of the price process $Q_t B_f(t,T)$ as a numéraire asset. We are now in a position to state the following lemma.

Lemma 4.4. *The price of the currency call option satisfies*

$$C_t(T,K) = Q_t B_f(t,T)\,\widehat{\mathbb{P}}_T\big(Q_T > K \,|\, \mathcal{F}_t\big) - K B_d(t,T)\,\mathbb{P}_T\big(Q_T > K \,|\, \mathcal{F}_t\big)$$

or, equivalently,

$$C_t(T,K) = Q_t B_f(t,T)\,\widehat{\mathbb{P}}_T\big(x_T > \ln K \,|\, \mathcal{F}_t\big) - K B_d(t,T)\,\mathbb{P}_T\big(x_T > \ln K \,|\, \mathcal{F}_t\big). \tag{14}$$

To complete the proof of Theorem 4.1, it remains to evaluate the conditional probabilities arising in formula (14). By another application of Girsanov's theorem, one can check that the process $(Q, v, r, \widehat{r})$ has the Markov property under the probability measures $\mathbb{P}_T$ and $\widehat{\mathbb{P}}_T$. In view of Proposition 3.1 and Lemma 3.1, the random variable x_T is a function of Q_T, r_T and $\widehat{r}_T$. We thus conclude that

$$C_t(T,K) = Q_t B_f(t,T)\, P_1(t, Q_t, v_t, r_t, \widehat{r}_t, K) - K B_d(t,T) P_2(t, Q_t, v_t, r_t, \widehat{r}_t, K) \tag{15}$$

where we denote

$$\begin{aligned} P_1(t, Q_t, v_t, r_t, \widehat{r}_t, K) &= \widehat{\mathbb{P}}_T(x_T > \ln K \,|\, Q_t, v_t, r_t, \widehat{r}_t),\\ P_2(t, Q_t, v_t, r_t, \widehat{r}_t, K) &= \mathbb{P}_T(x_T > \ln K \,|\, Q_t, v_t, r_t, \widehat{r}_t). \end{aligned}$$

To obtain explicit formulae for the conditional probabilities above, it suffices to derive the corresponding conditional characteristic functions

$$\begin{aligned} f_1(\phi, t, Q_t, v_t, r_t, \widehat{r}_t) &= \mathbb{E}_t^{\widehat{\mathbb{P}}_T}\big[\exp(i\phi x_T)\big],\\ f_2(\phi, t, Q_t, v_t, r_t, \widehat{r}_t) &= \mathbb{E}_t^{\mathbb{P}_T}\big[\exp(i\phi x_T)\big]. \end{aligned}$$

The idea is to use the Radon-Nikodým derivatives in order to obtain convenient expressions for the characteristic functions in terms of conditional expectations under the domestic spot martingale measure $\mathbb{P}$. The following lemma will allow us to achieve this goal.

Lemma 4.5. *The following equality holds*

$$\frac{d\widehat{\mathbb{P}}_T}{d\mathbb{P}}\Big|_{\mathcal{F}_t} = \exp\bigg(\int_0^t \sqrt{v_u}\,dW_u^Q - \int_0^t \sigma_f n_f(u,T)\sqrt{\widehat{r}_u}\,dW_u^f - \frac{1}{2}\int_0^t \big(v_u + \sigma_f^2 n_f^2(u,T)\widehat{r}_u\big)\,du\bigg).$$

Proof. Straightforward computations show that

$$\begin{aligned} \frac{d\widehat{\mathbb{P}}_T}{d\mathbb{P}}\Big|_{\mathcal{F}_t} &= \frac{d\widehat{\mathbb{P}}_T}{d\mathbb{P}_T}\Big|_{\mathcal{F}_t}\frac{d\mathbb{P}_T}{d\mathbb{P}}\Big|_{\mathcal{F}_t}\\ &= \exp\bigg(\int_0^t \widetilde{\sigma}_F(u,T)\cdot d\widetilde{W}_u^T - \frac{1}{2}\int_0^t \|\widetilde{\sigma}_F(u,T)\|^2\,du\bigg)\\ &\quad\times\ \exp\bigg(-\int_0^t \sigma_d n_d(u,T)\sqrt{r_u}\,dW_u^d - \frac{1}{2}\int_0^t \sigma_d^2 n_d^2(u,T) r_u\,du\bigg)\\ &= \exp\bigg(\int_0^t \sqrt{v_u}\,dW_u^Q + \int_0^t \sigma_d n_d(u,T)\sqrt{r_u}\,dW_u^T - \int_0^t \sigma_f n_f(u,T)\sqrt{\widehat{r}_u}\,dW_u^f\bigg)\\ &\quad\times\ \exp\bigg(-\frac{1}{2}\int_t^T \big(v_u + \sigma_d^2 n_d^2(u,T) r_u + \sigma_f^2 n_f^2(u,T)\widehat{r}_u\big)\,du\bigg)\\ &\quad\times\ \exp\bigg(-\int_0^t \sigma_d n_d(u,T)\sqrt{r_u}\,dW_u^d - \frac{1}{2}\int_0^t \sigma_d^2 n_d^2(u,T) r_u\,du\bigg). \end{aligned}$$

Using (12), we now obtain

$$\frac{d\widehat{\mathbb{P}}_T}{d\mathbb{P}}\Big|_{\mathcal{F}_t} = \exp\bigg(\int_0^t \sqrt{v_u}\, dW_u^Q - \int_0^t \sigma_f n_f(u,T)\sqrt{\widehat{r}_u}\, dW_u^f - \frac{1}{2}\int_0^t \big(v_u + \sigma_f^2 n_f^2(u,T)\widehat{r}_u\big)du\bigg),$$

which is the desired expression. ☐

In view of the formula established in Lemma 4.5 and the abstract Bayes formula, to compute $f_1(\phi) = f_1(\phi, t, Q_t, v_t, r_t, \widehat{r}_t)$, it suffices to focus on the following conditional expectation under $\mathbb{P}$

$$\begin{aligned} f_1(\phi) = \mathbb{E}_t^{\mathbb{P}}\bigg\{ & \exp\big(i\phi x_T\big) \exp\bigg(\int_t^T \sqrt{v_u}\, dW_u^Q - \int_t^T \sigma_f n_f(u,T)\sqrt{\widehat{r}_u}\, dW_u^f \\ & - \frac{1}{2}\int_t^T \big(v_u + \sigma_f^2 n_f^2(u,T)\widehat{r}_u\big)\, du\bigg)\bigg\}. \end{aligned} \tag{16}$$

Similarly, in view of formula (11), we obtain for $f_2(\phi) = f_2(\phi, t, Q_t, v_t, r_t, \widehat{r}_t)$

$$f_2(\phi) = \mathbb{E}_t^{\mathbb{P}}\bigg\{ \exp(i\phi x_T) \exp\bigg[- \int_t^T \sigma_d n_d(u,T)\sqrt{r_u}\, dW_u^d - \frac{1}{2}\int_t^T \sigma_d^2 n_d^2(u,T) r_u\, du\bigg]\bigg\}. \tag{17}$$

To proceed, we will need the following result, which is an immediate consequence of Lemma 4.3.

Corollary 4.1. *Under Assumptions (A.1)–(A.4), the process* $x_t = \ln F(t,T)$ *admits the following representation under the domestic forward martingale measure* $\mathbb{P}_T$

$$x_T = x_t + \int_t^T \widetilde{\sigma}_F(u,T)\cdot d\widetilde{W}_u^T - \frac{1}{2}\int_t^T \|\widetilde{\sigma}_F(u,T)\|^2\, du + J^Q(t,T) - \lambda_Q \mu_Q (T-t) \tag{18}$$

or, more explicitly,

$$\begin{aligned} x_T = x_t & + \int_t^T \sqrt{v_u}\, dW_u^Q + \int_t^T \sigma_d n_d(u,T)\sqrt{r_u}\, dW_u^T - \int_t^T \sigma_f n_f(u,T)\sqrt{\widehat{r}_u}\, dW_u^f \\ & - \frac{1}{2}\int_t^T \big(v_u + \sigma_d^2 n_d^2(u,T) r_u + \sigma_f^2 n_f^2(u,T)\widehat{r}_u\big)\, du \\ & + \sum_{k=N_t^Q+1}^{N_T^Q} \ln(1+J_k^Q) - \lambda_Q\mu_Q(T-t). \end{aligned}$$

Using equality (16) and Corollary 4.1, we obtain

$$\begin{aligned} f_1(\phi) \quad = \quad \mathbb{E}_t^{\mathbb{P}}\bigg\{ & \exp\big(i\phi x_T\big) \exp\bigg[\int_t^T \sqrt{v_u}\, dW_u^Q - \int_t^T \sigma_f n_f(u,T)\sqrt{\widehat{r}_u}\, dW_u^f \\ & - \frac{1}{2}\int_t^T \big(v_u + \sigma_f^2 n_f^2(u,T)\widehat{r}_u\big)\, du\bigg]\bigg\} \end{aligned}$$

so that

$$\begin{aligned} f_1(\phi) \quad = \quad \mathbb{E}_t^{\mathbb{P}}\bigg\{ & \exp\bigg[i\phi\bigg(x_t + \int_t^T \sqrt{v_u}\, dW_u^Q + \int_t^T \sigma_d n_d(u,T)\sqrt{r_u}\, dW_u^T - \int_t^T \sigma_f n_f(u,T)\sqrt{\widehat{r}_u}\, dW_u^f\bigg)\bigg] \\ & \times \exp\bigg[-\frac{i\phi}{2}\int_t^T \big(v_u + \sigma_d^2 n_d^2(u,T) r_u + \sigma_f^2 n_f^2(u,T)\widehat{r}_u\big)\, du\bigg] \\ & \times \exp\bigg[\int_t^T \sqrt{v_u}\, dW_u^Q - \int_t^T \sigma_f n_f(u,T)\sqrt{\widehat{r}_u}\, dW_u^f\bigg] \\ & \times \exp\bigg[-\frac{1}{2}\int_t^T \big(v_u + \sigma_f^2 n_f^2(u,T)\widehat{r}_u\big)\, du\bigg]\bigg\} \\ & \times \exp\Big[i\phi J^Q(t,T) - i\phi\lambda_Q\mu_Q(T-t)\Big]. \end{aligned}$$

For the sake of conciseness, we denote $\alpha = 1 + i\phi$, $\beta = i\phi$ and $c_t = \exp(i\phi x_t)$. After simplifications and rearrangement, the formula above becomes

$$\begin{aligned} f_1(\phi) &= c_t\,\mathbb{E}_t^{\mathbb{P}}\Bigg\{ \exp\left[\alpha\Big(\int_t^T \sqrt{v_u}\,dW_u^Q - \frac{1}{2}\int_t^T v_u\,du\Big)\right] \\ &\quad \times \exp\left[\beta\Big(\int_t^T \sigma_d n_d(u,T)\sqrt{r_u}\,dW_u^T - \frac{1}{2}\int_t^T \sigma_d^2 n_d^2(u,T) r_u\,du\Big)\right] \\ &\quad \times \exp\left[-\alpha\Big(\int_t^T \sigma_f n_f(u,T)\sqrt{\widehat{r}_u}\,dW_u^f + \frac{1}{2}\int_t^T \sigma_f^2 n_f^2(u,T)\widehat{r}_u\,du\Big)\right] \\ &\quad \times \exp\Big[\beta J^Q(t,T) - \beta\lambda_Q\mu_Q(T-t)\Big]\Bigg\}. \end{aligned}$$

In view of Assumptions (A.1)–(A.6), we may use the following representation for the Brownian motion W^Q

$$W_t^Q = \rho\, W_t^v + \sqrt{1-\rho^2}\, W_t \tag{19}$$

where $W = (W_t)_{t\in[0,T]}$ is a Brownian motion under $\mathbb{P}$ independent of the Brownian motions W^v, W^d and W^f. Consequently, the conditional characteristic function $f_1(\phi)$ can be represented in the following way

$$\begin{aligned} f_1(\phi) &= c_t\,\mathbb{E}_t^{\mathbb{P}}\Bigg\{ \exp\left[\alpha\rho\int_t^T \sqrt{v_u}\,dW_u^v + \alpha\sqrt{1-\rho^2}\int_t^T \sqrt{v_u}\,dW_u - \frac{\alpha}{2}\int_t^T v_u\,du\right] \\ &\quad \times \exp\left[\beta\Big(\int_t^T \sigma_d n_d(u,T)\sqrt{r_u}\,dW_u^T - \frac{1}{2}\int_t^T \sigma_d^2 n_d^2(u,T) r_u\,du\Big)\right] \\ &\quad \times \exp\left[-\alpha\Big(\int_t^T \sigma_f n_f(u,T)\sqrt{\widehat{r}_u}\,dW_u^f + \frac{1}{2}\int_t^T \sigma_f^2 n_f^2(u,T)\widehat{r}_u\,du\Big)\right] \\ &\quad \times \exp\Big[\beta J^Q(t,T) - \beta\lambda_Q(T-t)\mu_Q\Big]\Bigg\}. \end{aligned} \tag{20}$$

By combining Proposition 3.1 with Definition 4.1, we obtain the following auxiliary result, which will be helpful in the proof of Theorem 4.1.

Lemma 4.6. *Given the dynamics (1) of processes v, r and $\widehat{r}$ and formula (12), we obtain the following equalities*

$$\begin{aligned} &\int_t^T \sqrt{v_u}\,dW_u^v = \frac{1}{\sigma_v}\Big(v_T - v_t - \theta\tau + \kappa\int_t^T v_u\,du - (Z_T^v - Z_t^v)\Big), \\ &\int_t^T \sigma_d n_d(u,T)\sqrt{r_u}\,dW_u^T - \frac{1}{2}\int_t^T \sigma_d^2 n_d^2(u,T) r_u\,du = -n_d(t,T)r_t - \int_t^T a_d n_d(u,T)\,du \\ &\qquad\qquad + \int_t^T r_u\,du, \\ &\int_t^T \sigma_f n_f(u,T)\sqrt{\widehat{r}_u}\,dW_u^f + \frac{1}{2}\int_t^T \sigma_f^2 n_f^2(u,T)\widehat{r}_u\,du = -n_f(t,T)\widehat{r}_t - \int_t^T a_f n_f(u,T)\,du \\ &\qquad\qquad + \int_t^T \widehat{r}_u\,du. \end{aligned}$$

Proof. The first asserted formula is an immediate consequence of (1). For the second, we recall that the function $n_d(t,T)$ is known to satisfy the following differential equation, for any fixed $T > 0$,

$$\frac{\partial n_d(t,T)}{\partial t} - \frac{1}{2}\sigma_d^2 n_d^2(t,T) - b_d n_d(t,T) + 1 = 0$$

with the terminal condition $n_d(T,T) = 0$. Therefore, using the Itô formula and equality (12), we obtain

$$\begin{aligned}
d(n_d(t,T)r_t) &= r_t\, dn_d(t,T) + n_d(t,T)\, dr_t \\
&= r_t\Big(\frac{1}{2}\,\sigma_d^2 n_d^2(t,T) + b_d n_d(t,T) - 1\Big)\, dt + n_d(t,T)(a_d - b_d r_t)\, dt + n_d(t,T)\sigma_d\sqrt{r_t}\, dW_t^d \\
&= \frac{1}{2}\,\sigma_d^2 n_d^2(t,T) r_t\, dt - r_t\, dt + n_d(t,T)a_d\, dt + n_d(t,T)\sigma_d\sqrt{r_t}\, dW_t^d \\
&= -\frac{1}{2}\,\sigma_d^2 n_d^2(t,T) r_t\, dt - r_t\, dt + n_d(t,T)a_d\, dt + n_d(t,T)\sigma_d\sqrt{r_t}\, dW_t^T.
\end{aligned}$$

This yields the second asserted formula, upon integration between t and T. The derivation of the last one is based on the same arguments and thus it is omitted. □

4.2 Proof of Theorem 4.1

The proof of Theorem 4.1 is split into two steps in which we deal with $f_1(\phi)$ and $f_2(\phi)$, respectively.

Step 1. We will first compute $f_1(\phi)$. By combining (20) with the equalities derived in Lemma 4.6, we obtain the following representation for $f_1(\phi)$

$$\begin{aligned}
f_1(\phi) \quad = \quad & c_t\,\mathbb{E}_t^{\mathbb{P}}\Bigg\{\exp\Bigg[-\frac{\alpha\rho}{\sigma_v}(v_t+\theta\tau) + \Big(\frac{\alpha\rho\kappa}{\sigma_v} - \frac{\alpha}{2}\Big)\int_t^T v_u\, du \\
& + \alpha\sqrt{1-\rho^2}\int_t^T \sqrt{v_u}\, dW_u - \frac{\alpha\rho}{\sigma_v}(Z_T^v - Z_t^v) + \frac{\alpha\rho}{\sigma_v}\,v_T\Bigg] \\
& \times \exp\Bigg[-\beta\Big(n_d(t,T)r_t + \int_t^T a_d n_d(u,T)\, du\Big) + \beta\int_t^T r_u\, du\Bigg] \\
& \times \exp\Bigg[\alpha\Big(n_f(t,T)\widehat{r}_t + \int_t^T a_f n_f(u,T)\, du\Big) - \alpha\int_t^T \widehat{r}_u\, du\Bigg] \\
& \times \exp\Bigg[\beta J^Q(t,T) - \beta\lambda_Q\mu_Q(T-t) - \frac{\alpha\rho}{\sigma_v}(Z_T^v - Z_t^v)\Bigg]\Bigg\}.
\end{aligned}$$

Recall the well-known property that if ζ has the standard normal distribution then $\mathbb{E}(e^{z\zeta}) = e^{z^2/2}$ for any complex number $z \in \mathbb{C}$.

Consequently, by conditioning first on the sample path of the process $(v, r, \widehat{r})$ and using the independence of the processes $(v, r, \widehat{r})$ and W under $\mathbb{P}$ and Lemma 4.1, we obtain

$$\begin{aligned}
f_1(\phi) \quad = \quad & c_t \exp\Big[\lambda_Q\tau\Big((1+\mu_Q)^\beta e^{-\frac{1}{2}\beta\gamma\sigma_Q^2} - 1\Big)\Big] \\
& \times \exp\Big[-\Big(\beta\lambda_Q\mu_Q\tau + \frac{\alpha\rho}{\sigma_v}(v_t+\theta\tau)\Big)\Big] \\
& \times \exp\Big[\lambda_v\sigma_v\tau\Big(\frac{p\eta_1}{\sigma_v\eta_1+\alpha\rho} + \frac{q\eta_2}{\sigma_v\eta_2-\alpha\rho} - 1\Big)\Big] \\
& \times \exp\Big[-\beta\Big(n_d(t,T)r_t + \int_t^T a_d n_d(u,T)\, du\Big)\Big] \\
& \times \exp\Big[\alpha\Big(n_f(t,T)\widehat{r}_t + \int_t^T a_f n_f(u,T)\, du\Big)\Big] \\
& \times \mathbb{E}_t^{\mathbb{P}}\Bigg\{\exp\Bigg[\frac{\alpha\rho}{\sigma_v}v_T + \Big(\frac{\alpha^2(1-\rho^2)}{2} + \frac{\alpha\rho\kappa}{\sigma_v} - \frac{\alpha}{2}\Big)\int_t^T v_u\, du\Bigg] \\
& \times \exp\Bigg[\beta\int_t^T r_u\, du - \alpha\int_t^T \widehat{r}_u\, du\Bigg]\Bigg\}.
\end{aligned}$$

This in turn implies that the following equality holds

$$\begin{aligned}
f_1(\phi) &= c_t \exp\left[\lambda_Q\tau\left((1+\mu_Q)^\beta e^{-\frac{1}{2}\beta\gamma\sigma_Q^2}-1\right)\right] \\
&\quad\times \exp\left[-\left(\beta\lambda_Q\mu_Q\tau+\frac{\alpha\rho}{\sigma_v}(v_t+\theta\tau)\right)\right] \\
&\quad\times \exp\left[\lambda_v\sigma_v\tau\left(\frac{p\eta_1}{\sigma_v\eta_1+\alpha\rho}+\frac{q\eta_2}{\sigma_v\eta_2-\alpha\rho}-1\right)\right] \\
&\quad\times \exp\left[-\beta\left(n_d(t,T)r_t+\int_t^T a_d n_d(u,T)\,du\right)\right] \\
&\quad\times \exp\left[\alpha\left(n_f(t,T)\widehat{r}_t+\int_t^T a_f n_f(u,T)\,du\right)\right] \\
&\quad\times \mathbb{E}_t^{\mathbb{P}}\left\{\exp\left[-s_1v_T-s_2\int_t^T v_u\,du-s_3r_T-s_4\int_t^T r_u\,du-s_5\widehat{r}_T-s_6\int_t^T\widehat{r}_u\,du\right]\right\}
\end{aligned}$$

where the constants s_1,s_2,s_3,s_4,s_5,s_6 are given by (7). A direct application of Lemma 4.2 furnishes an explicit formula for $f_1(\phi)$, as reported in the statement of Theorem 4.1.

Step 2. In order to compute the conditional characteristic function

$$f_2(\phi)=f_2(\phi,t,Q_t,v_t,r_t,\widehat{r}_t)=\mathbb{E}_t^{\mathbb{P}_T}\left[\exp(i\phi x_T)\right]$$

we proceed in an analogous manner as for $f_1(\phi)$. We first recall that (see (17))

$$f_2(\phi)=\mathbb{E}_t^{\mathbb{P}}\left\{\exp(i\phi x_T)\exp\left[-\int_t^T\sigma_d n_d(u,T)\sqrt{r_u}\,dW_u^d-\frac{1}{2}\int_t^T\sigma_d^2n_d^2(u,T)r_u\,du\right]\right\}.$$

Therefore, using Corollary 4.1, we obtain

$$\begin{aligned}
f_2(\phi) &= c_t\,\mathbb{E}_t^{\mathbb{P}}\left\{\exp\left[i\phi\left(\int_t^T\sqrt{v_u}\,dW_u^Q+\int_t^T\sigma_d n_d(u,T)\sqrt{r_u}\,dW_u^T-\int_t^T\sigma_f n_f(u,T)\sqrt{\widehat{r}_u}\,dW_u^f\right)\right]\right. \\
&\quad\times \exp\left[-i\phi\left(\frac{1}{2}\int_t^T\left(v_u+\sigma_d^2n_d^2(u,T)r_u+\sigma_f^2n_f^2(u,T)\widehat{r}_u\right)du\right)\right] \\
&\quad\left.\times \exp\left[-\int_t^T\sigma_d n_d(u,T)\sqrt{r_u}\,dW_u^d-\frac{1}{2}\int_t^T\sigma_d^2n_d^2(u,T)r_u\,du\right]\times\exp\left[i\phi J^Q(t,T)\right]\right\}.
\end{aligned}$$

Consequently, using formulae (12), (19) and Lemma 4.1, we obtain the following expression for $f_2(\phi)$

$$\begin{aligned}
f_2(\phi) &= c_t\,\exp\left[\lambda_Q\tau\left((1+\mu_Q)^\beta e^{-\frac{1}{2}\beta\gamma\sigma_Q^2}-1\right)-\beta\lambda_Q\mu_Q\tau\right] \\
&\quad\times \mathbb{E}_t^{\mathbb{P}}\left\{\exp\left[\beta\left(\rho\int_t^T\sqrt{v_u}\,dW_u^v+\sqrt{1-\rho^2}\int_t^T\sqrt{v_u}\,dW_u-\int_t^T\sigma_f n_f(u,T)\sqrt{\widehat{r}_u}\,dW_u^f\right)\right]\right. \\
&\quad\times \exp\left[-\beta\left(\frac{1}{2}\int_t^T\left(v_u+\sigma_f^2n_f^2(u,T)\widehat{r}_u\right)du\right)\right] \\
&\quad\left.\times \exp\left[-\gamma\left(\int_t^T\sigma_d n_d(u,T)\sqrt{r_u}\,dW_u^d+\frac{1}{2}\int_t^T\sigma_d^2n_d^2(u,T)r_u\,du\right)\right]\right\}.
\end{aligned}$$

Similarly as in the case of $f_1(\phi)$, we condition on the sample path of the process $(v,r,\widehat{r})$ and we use the postulated independence of the processes $(v,r,\widehat{r})$ and W under $\mathbb{P}$. By invoking also Lemma 4.1, we obtain

$$\begin{aligned} f_2(\phi) &= c_t \exp\left[\lambda_Q \tau \left((1+\mu_Q)^\beta e^{-\frac{1}{2}\beta\gamma\sigma_Q^2} - 1\right) - \beta\lambda_Q\mu_Q\tau\right] \\ &\times \mathbb{E}_t^{\mathbb{P}}\Bigg\{ \exp\left[\beta\rho \int_t^T \sqrt{v_u}\, dW_u^v + \frac{\beta^2(1-\rho^2)-\beta}{2} \int_t^T v_u\, du\right] \\ &\times \exp\left[-\gamma\Big(\int_t^T \sigma_d n_d(u,T)\sqrt{r_u}\, dW_u^d + \frac{1}{2}\int_t^T \sigma_d^2 n_d^2(u,T) r_u\, du\Big)\right] \\ &\times \exp\left[-\beta\Big(\int_t^T \sigma_f n_f(u,T)\sqrt{\widehat{r}_u}\, dW_u^f + \frac{1}{2}\int_t^T \sigma_f^2 n_f^2(u,T)\widehat{r}_u\, du\Big)\right]\Bigg\}. \end{aligned}$$

Using Lemma 4.6, we conclude that

$$\begin{aligned} f_2(\phi) &= c_t \exp\left[\lambda_Q\tau\left((1+\mu_Q)^\beta e^{-\frac{1}{2}\beta\gamma\sigma_Q^2} - 1\right)\right] \\ &\times \exp\left[-\left(\beta\lambda_Q\mu_Q\tau + \frac{\beta\rho}{\sigma_v}(v_t+\theta\tau)\right)\right] \\ &\times \exp\left[\lambda_v\sigma_v\tau\left(\frac{p\eta_1}{\sigma_v\eta_1+\beta\rho} + \frac{q\eta_2}{\sigma_v\eta_2-\beta\rho} - 1\right)\right] \\ &\times \exp\left[\gamma\left(n_d(t,T)r_t + \int_t^T a_d n_d(u,T)\, du\right)\right] \\ &\times \exp\left[\beta\left(n_f(t,T)\widehat{r}_t + \int_t^T a_f n_f(u,T)\, du\right)\right] \\ &\times \mathbb{E}_t^{\mathbb{P}}\left\{\exp\left[-q_1 v_T - q_2\int_t^T v_u\, du - q_3 r_T - q_4\int_t^T r_u\, du - q_5\widehat{r}_T - q_6\int_t^T \widehat{r}_u\, du\right]\right\} \end{aligned}$$

with the coefficients $q_1, q_2, q_3, q_4, q_5, q_6$ reported in formula (8). Another straightforward application of Lemma 4.2 yields the closed-form expression (6) for the conditional characteristic function $f_2(\phi)$.

To complete the proof of Theorem 4.1, it suffices to combine formula (15) with the standard inversion formula (4) providing integral representations for the conditional probabilities

$$P_1(t, Q_t, v_t, r_t, \widehat{r}_t, K) = \widehat{\mathbb{P}}_T(x_T > \ln K \,|\, Q_t, v_t, r_t, \widehat{r}_t)$$

and

$$P_2(t, Q_t, v_t, r_t, \widehat{r}_t, K) = \mathbb{P}_T(x_T > \ln K \,|\, Q_t, v_t, r_t, \widehat{r}_t).$$

This ends the derivation of the pricing formula for the foreign exchange call option. The price of the corresponding put option is readily available as well, due to the put-call parity relationship for currency options (see formula (22) in Section 5). □

5 Numerical Results

The goal of the final section is to illustrate our approach by means of numerical examples in which we apply our FX market model, that is, the Heston/CIR double exponential jump-diffusion model, and we compare this approach with other related models that were recently proposed in Moretto et al. (2010) and Ahlip and Rutkowski (2013) to deal with the exchange rate derivatives.

5.1 FX Market Conventions

Let us start by noting that the foreign exchange market differs from equity markets, since the market quotes for options are not made directly for a family strikes. Indeed, the currency option prices are typically quoted in terms of the associated implied volatilities for a given time to expiry $\tau = T - t$ and a fixed value of the *forward delta* Δ_F, that is, a fixed hedge ratio in the forward FX market. Other possible conventions are based on the *spot delta*, which refers to a hedge in the spot market, or the *premium-adjusted delta*, which refers to the case where the option's premium is paid in the foreign currency. All these conventions for market quotes of currency options are based on the explicit pricing formula obtained for the classic Garman-Kohlhagen lognormal model of the exchange rate dynamics (see, for instance, Section 4.2 in Musiela and Rutkowski (2005)).

From the Garman-Kohlhagen pricing formula it follows, in particular, that for a given forward delta Δ_F and a quoted implied volatility σ, the corresponding strike price $K = K(\Delta_F)$ is given by the following conversion formula

$$K(\Delta_F) = F(t,T)\, \exp\Big(- c\sigma\sqrt{\tau}\, N^{-1}(c\Delta_F) + \frac{1}{2}\sigma^2\tau\Big) \tag{21}$$

where $\tau = T - t$, we denote by N^{-1} the inverse of the standard normal cumulative distribution function, and the marker c satisfies: $c = 1$ ($c = -1$, resp.) for the call (put, resp.) currency option. In view of the conversion formula (21), market quotation of option prices based on the implied volatility for fixed forward deltas is in fact equivalent to quoting prices for the corresponding fixed strikes.

According to commonly adopted terminology, the call option with the forward delta $\Delta_F = 25\%$ is called the *25-delta call*. Similarly, the put option corresponding to the forward delta $\Delta_F = -25\%$ is referred to as the *25-delta put*. One of the reasons why the forward deltas are often used in volatility quotes for FX options is the fact that the forward delta of a call and the absolute value of the delta of the corresponding put add up to 1 so that, for instance, a 25-delta call (resp., a 75-delta call) must have the same implied volatility as a 75-delta put (resp., a 25-delta put) according to this convention.

Another important feature is that the currency derivatives are based on the notion of the *at-the-money forward* (ATMF) exchange rate, that is, the forward exchange rate $F(t,T)$ obtained by exploiting the interest rate parity implicit in equation (3). It can be checked that the forward delta of the ATMF call option is not equal exactly to 0.5 and thus the ATMF call does not coincide with the 50-delta call. Recall also that the universal put-call parity formula for plain-vanilla currency options reads

$$C_t(T,K) - P_t(T,K) = Q_t B_f(t,T) - K B_d(t,T) \tag{22}$$

where $C_t(T,K)$ and $P_t(T,K)$ are prices of the currency call and put options, respectively. In particular, the prices of ATMF call and put options are equal in any arbitrage-free market model; for this reason, the ATMF rate is also known as the *ATM-value-neutral* strike.

An alternative convention is to use the *ATM-delta-neutral* strike, that is, the level of the strike K for which the absolute values of forward deltas of call and put options coincide, so that they are both equal to 0.5. This is in fact the prevailing convention for the currency options and thus henceforth by the *ATM call* we will mean the ATM-delta-neutral call. For a more detailed discussion of foreign market conventions, the interested reader may consult, for instance, Clark (2011), Hakala and Wystup (2002) or Reiswich and Wystup (2010, 2012).

By convention, we assume hereafter that USD is the domestic currency, whereas EUR is the foreign currency. Note that the EUR/USD exchange rate is the price of one Euro in US dollars. Equation (21) is suitable in the case of EUR/USD option where the premium is paid in US dollars, that is, the domestic currency. It would be incorrect, for example, for USD/JPY option where the premium is also paid in US dollars – indeed, in that case the market convention based on the premium-adjusted delta would be more suitable.

5.2 Market Data

In our numerical results presented in what follows, we make use (with the kind permission of the authors) of the data for the EUR/USD options and bond yields from the paper by Moretto et al. (2010) (see page 469 therein). Note that positive (resp. negative) values of the forward delta Δ_F

correspond to market quotes for call (resp. put) options. The EUR/USD spot exchange rate on June 13, 2005 was $Q_0 = 1.2087$.

Δ_F	-10%	-15%	-25%	ATM (50%)	25%	15%	10%
1M	10.36%	10.09%	9.73%	9.30%	9.15%	9.18%	9.25%
2M	10.28%	10.01%	9.65%	9.25%	9.15%	9.22%	9.31%
3M	10.22%	9.95%	9.62%	9.25%	9.19%	9.28%	9.39%
6M	10.23%	9.95%	9.64%	9.35%	9.39%	9.55%	9.74%
9M	10.22%	9.96%	9.96%	9.40%	9.49%	9.68%	9.88%
1Y	10.24%	9.98%	9.69%	9.45%	9.56%	9.77%	9.99%

Table 1: Market implied volatilities for EUR/USD options on June 13, 2005 (original source of data: Banca Caboto S.p.A. – Gruppo Intesa, Milano).

Rates	r_d	r_f
1M	3.14%	2.09%
2M	3.22%	2.09%
3M	3.32%	2.10%
6M	3.50%	2.09%
9M	3.60%	2.09%
1Y	3.68%	2.09%

Table 2: Market domestic (USD) and foreign (EUR) interest rates on June 13, 2005 (original source of data: Banca Caboto S.p.A. – Gruppo Intesa, Milano).

5.3 Comparison of Model Prices

We provide a comparison of option prices for the foreign exchange version of Heston's model, the Heston/CIR (HCIR) model examined in Ahlip and Rutkowski (2013), and the Heston/CIR/ Double Exponential Jump-Diffusion (HCIR–DEJD) model put forward in this paper. According to equation (1), the dynamics of the exchange rate Q and its squared volatility v involve seven parameters, which are listed in Table 3. In addition, one needs to specify three parameters for each of the interest rates, r_d and r_f. For each maturity date and delta, the initial values of the volatilities v_0, which are the square of the implied volatilities from Table 1, are listed in 5 (this particular choice is suggested by formula (5.1) in D'Ippoliti *et al.* (2010)).

It should be acknowledged that we do not attempt to calibrate the three alternative models compared in this paper to market data. We limit ourselves to illustrating the impact of additional components of a model on prices of long-date options, and thus our choice of parameters is somewhat arbitrary.

θ	κ	η_1	η_2	λ_v	p	ρ
0.02606	0.091	25	25	3	0.5	0.9786

Table 3: Parameter values for the exchange rate and its squared volatility.

The parameter values for the CIR dynamics of domestic and foreign interest rates are given in Table 4. It is fair to say that the choice of these parameters was rather artificial and made for illustrative purposes only. The initial values of the domestic and foreign short-term rates were inferred by matching, independently for each maturity date, the zero-coupon bond yield given in Table 2 with the bond pricing formula from Proposition 3.1. In the case of Heston's model, the bond yield coincides with the constant interest rate for each particular maturity. Needless to say that we use the same values of parameters for all models, although not all parameters are actually employed in every model.

a_d	b_d	σ_d	a_f	b_f	σ_f
0.0332	0.03	0.25	0.021	0.024	0.24

Table 4: Parameter values for domestic and foreign interest rates.

In Table 5, we report prices of ATM calls for expiries ranging from one month to one year. We use here the ATM volatilities for different maturities, as given in Table 1, and the interest rates from Table 2. In all examples reported here, the strikes are computed using the conversion formula (21). We observe that the prices obtained using the HCIR–DEJD model are substantially higher than the prices obtained for the Heston and HCIR models, especially for options with longer maturities. Although we only deal here with plain-vanilla currency options, our numerical results support the conjecture that jumps in the volatility dynamics and the uncertain character of interest rates are likely to play an important role in valuation of long-dated hybrid foreign exchange derivatives, such as PRDCs or FX-TARNs.

Maturity	Initial Vols	Strike	Heston price	HCIR price	HCIR–DEJD price
1M	0.008649	1.21019	0.0134635	0.0134861	0.0134874
2M	0.008556	1.21184	0.0198044	0.0198958	0.0199012
3M	0.008556	1.21369	0.0252715	0.0254794	0.0254927
6M	0.008742	1.21991	0.0399683	0.0408467	0.0408692
9M	0.008836	1.22652	0.0533947	0.0537438	0.0557655
1Y	0.009025	1.23357	0.0664330	0.0676871	0.0714555

Table 5: Prices of ATM call options using data of June 13, 2005.

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